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ABSTRACT

In the literature a belief is spreading that the static electrovac counterparts of the Tomimatsu-Satō solutions are known. However, as we show, the counterpart metrics are obtained by means of a wrong method, and do not describe electrovac fields. In this paper we give the true static electrovac counterparts.

АННОТАЦИЯ

В научной литературе известны решения для статичного электровакуума, аналогичные решению Томимацу-Сато уравнения Эйнштейна. В настоящей работе доказано, что метод получения этого решения неправильный; описан правильный метод и соответствующие правильные решения для статичного электровакуума.

KIVONAT

Ismeretes az irodalomban egy eljárás a Tomimatsu-Satō megoldások statikus elektrovákuum megfelelőinek előállítására. E cikkben megmutatjuk, hogy e módszer hibás, és megadjuk a helyes eljárást és a valódi statikus elektrovákuum megoldásokat.

I. INTRODUCTION

Bonnor has given a method for constructing static electrovac solutions of the Einstein equation from stationary vacuum ones[1] and has calculated the counterpart of the Kerr metric[2]. The Kerr metric is the first member of the stationary Tomimatsu-Satō /TS/ series[3] and it is obvious to look for the Bonnor counterparts of the whole series.

Misra, Pandey, Srivastava and Tripathi have given a similar method for the same problem[4], by means of which they claim to have constructed the electrovac counterpart of the Kerr metric too[4], and Wang has performed their procedure for the second and third TS metrics[5]. /The two metrics obtained from the Kerr solution are different; Bonnor's space-time has two point singularities on the symmetry axis, while the solution of Misra & al. has the same singularity structure as the Kerr one./ Thus in the literature a belief keeps spreading that the electrostatic counterparts of the TS series are known. In this paper we show that the solutions given by Misra & al. and Wang are not the static electrovac counterparts of the TS solutions, moreover, they are not electrovac solutions. In Sects. 3 and 4 we give the Bonnor counterparts of the series. First, let us recapitulate the results of Misra & al.

These authors investigate the static axisymmetric electrovac problem. First, they show that one of the two components of the electromagnetic potential can be made zero using a duality rotation /see also in Ref.6/. Here the nonvanishing component will be referred to as A_o, since this can be achieved by choosing the duality angle in an appropriate way. After introducing the complex quantity

$$E \equiv \sqrt{g_{oo}} + iA_o \qquad /1/$$

two of their field equations go over to Ernst's equation for a stationary axisymmetric vacuum[7]. /The corresponding quantity of Ernst is $\varepsilon=f+i\varphi$/. Thus their result is that there is an "accidental symmetry" between the static electrovac and stationary vacuum problems: if /in Ernst's notation/ the substitutions

$$\begin{aligned} \varphi &\to \phi \\ f^2 &\to f \end{aligned} \qquad /2/$$

are performed, a new solution is obtained.

However, Ernst's equations for the electrovac case[8] show that /2/ is not a symmetry; Bonnor's transformation /in this paper Bonnorification/ is a more sophisticated procedure than /2/. On the other hand, the field equations of Misra & al. have been correctly obtained from the Einstein equation with their energy-momentum tensor.

It is easy to understand the contradiction. In fact, there is an "accidental symmetry" with respect to the transformation /2/: Perjés has shown[9] that if one changes the sign of the gravitational constant, /2/ yields electrovac solutions. Thus this symmetry has no physical meaning. This suggests that the paper of Misra & al. contains a sign error in the Einstein equation, and, in fact, they have chosen a wrong sign in the energy-momentum tensor. In order to prove this, we do not want to refer to the literature where is no unique convention for the signature but show that their energy-momentum tensor does not fulfil the strong energy condition which is satisfied for electromagnetic field[10].

The signature in Ref. 4 is /+---/. Then the strong energy condition can be written as

$$(T_{ik} - \tfrac{1}{2}g_{ik}T)v^i v^k \geq 0 \qquad /3/$$

for any v^i if $v^r v^s g_{rs}=+1$

The energy-momentum tensor of Misra & al. is:

$$T_{ik} = \frac{1}{4\pi}(F_{ir}F^r_k - \tfrac{1}{4}g_{ik}F_{rs}F^{rs})$$
$$F_{ik} = -F_{ki} \qquad /4/$$

The inequality /3/ is a pure algebraic expression and has to be valid in every point. Choose a point and perform a coordinate transformation leading to a Minkowskian line element in that point; then the freedom of the Lorentz transformations still remains. Choosing an arbitrary v^i, we can go over to a comoving frame where $v^i=\delta^i_o$. Then condition /3/ is of the form

$$-\frac{1}{8\pi}\{\sum_{R=1}^{3}(F_{oR})^2 + \sum_{R,S=1}^{3} F_{RS}F_{RS}\} \geq 0 \qquad /5/$$

The expression in the bracket is positive definite, thus condition /3/ cannot be fulfilled except the vacuum case. Changing the sign of T_{ik}, /3/ becomes trivial. Thus the energy-momentum tensor of Ref. 4 is of the wrong sign, and though the results of Misra & al. are correct for this energy-momentum tensor, their field equations are not the field equations of the static electrovac problem, hence their solutions are not the electrovac counterparts of stationary vacuum metrics. Similarly, Wang's solutions are not electrovac metrics.

In this paper we give the method of Bonnorifying the TS solutions. The formulae will be valid for any member of the series. It is interesting that Wang's asymptotic expression for the electromagnetic field[5] remains valid.

II. THE GENERAL METHOD

Here we recapitulate the general method of Bonnorification: Take a stationary axially symmetric vacuum metric in the canonical form

$$ds^2 = f(dt + \omega d\psi)^2 - f^{-1}[e^{2\gamma}(d\rho^2+dz^2)+\rho^2 d\psi^2] \qquad /6/$$

where f, ω and γ do not depend on ψ and t. Using the Papapetrou scalar φ

$$\nabla\varphi = -\frac{1}{\rho}f^2 \hat{n} \times \nabla\omega \qquad /7/$$

/$\hat{n}$ is the azimuthal unit vector/, the following field equations can be obtained:

$$\begin{aligned} f\Delta f &= (\nabla f)^2 - (\nabla\varphi)^2 \\ f\Delta\varphi &= 2\nabla f\nabla\varphi \\ \gamma,_1/\rho &= \frac{1}{4f^2}(f,_1{}^2 - f,_2{}^2 + \varphi,_1{}^2 - \varphi,_2{}^2) \\ \gamma,_2/\rho &= \frac{1}{2f^2}(f,_1 f,_2 + \varphi,_1\varphi,_2) \\ x^1 &= \rho, \quad x^2 = z \end{aligned} \qquad /8/$$

where the background metric is flat $(d\sigma^2 = d\rho^2 + dz^2 + \rho^2 d\psi^2)$ and the integrability condition for γ is a consequence of the system /8/. Having obtained a solution of the first two equations, γ can be calculated by quadrature.

Next, the field equations for static axisymmetric electrovac will be written as

$$\begin{aligned} f\Delta f &= (\nabla f)^2 + 2f(\nabla\phi)^2 \\ f\Delta\phi &= \nabla f\nabla\phi \\ \gamma,_1/\rho &= \frac{1}{4f^2}(f,_1{}^2 - f,_2{}^2 - 4f\phi,_1{}^2 + 4f\phi,_2{}^2) \\ \gamma,_2/\rho &= \frac{1}{2f^2}(f,_1 f,_2 - 4f\phi,_1\phi,_2) \end{aligned} \qquad /9/$$

where ϕ is real /in this case it is equal to A_o/; the more general case can be obtained by means of a duality rotation $\phi \to \phi e^{iC}$, which does not change the line element.

It is seen that /2/ does not transform the first two of /8/ into the first two of /9/. The correct symmetry is the following[11].

$$\begin{aligned} &\varphi \to i\phi \\ &f^2 \to f \end{aligned} \qquad /10/$$

However, of course, f, φ and ϕ have to be real, thus eq, /10/ can be valid only for some complex extensions of f and φ. It is not obvious that such an extension is always possible, but in many cases the suitable complex solution can easily be obtained.

The essence of the method that f, φ and ϕ be real quantities in the eqs. /8/ and /9/. Thus Ernst's complex quantities ξ and q

$$\begin{aligned} \varepsilon &= f+i\varphi = \frac{\xi-1}{\xi+1} \\ \phi &= \frac{q}{\xi+1} \end{aligned} \qquad /11/$$

generally used for stationary axisymmetric electrovac fields, are not convenient for the Bonnorification.

III. BONNORIFICATION OF THE TS SOLUTIONS

Let us introduce the spheroidal coordinates x, y:

$$\rho = \frac{1}{c}(x^2-1)^{1/2}(1-y^2)^{1/2}; \quad z = \frac{1}{c}xy; \qquad /12/$$

Then the TS solutions are of form

$$\xi = \frac{\alpha_1^{(\delta)}(p,q^2;x,y^2)+iqy\alpha_2^{(\delta)}(p,q^2;x,y^2)}{\beta_1^{(\delta)}(p,q^2;x,y^2)+iqy\beta_2^{(\delta)}(p,q^2;x,y^2)}; \quad p\equiv\sqrt{1-q^2} \qquad /13/$$

where δ is a positive integer. α_1, α_2, β_1 and β_2 are polynomials in x and y^2; for these polynomials/8/ gives two equations[3], whose forms are very complicated, thus they will not be cited here. The paper of Tomimatsu & Sato[3] contains the first four solutions. For the sake of simplicity here we give only the first two:

δ	α_1	α_2	β_1	β_2
1	px	-1	1	0
/Kerr/				
2	$p^2x^4+q^2y^4-1$	$-2px(x^2-y^2)$	$2px(x^2-1)$	$-2(1-y^2)$

In the limit $x\to\infty$, $\alpha_1 \simeq p^\delta x^{\delta^2}$, $\beta_1 \simeq p^{\delta-1} x^{\delta^2-1}$.

The first step of Bonnorification is to construct f and φ.
From eqs. /11/ and /13/:

$$f = \frac{\alpha_1^2-\beta_1^2+q^2y^2(\alpha_2^2-\beta_2^2)}{(\alpha_1+\beta_1)^2+q^2y^2(\alpha_2+\beta_2)^2} \qquad /14/$$

$$\varphi = \frac{-2qy(\alpha_1\beta_2-\beta_1\alpha_2)}{(\alpha_1+\beta_1)^2+q^2y^2(\alpha_2+\beta_2)^2}$$

Since the coordinates have to remain real, the complex extensions of f and φ can be obtained by complexifying the parameter q, thus f and φ remain solutions of the first two of eqs. /8/. As α and β contain only even powers of q, thus f will remain real and φ becomes imaginary if we perform:

$$q \to iq$$
$$p \to \sqrt{1+q^2} \qquad /15/$$

This substitution will be denoted by an asterisk on α_i and β_i:

$$\alpha_1{}^* = \alpha_1(\sqrt{1+q^2}\ , -q^2;\ x,\ y^2),\ \text{etc.} \qquad /16/$$

The transformation /10/ gives the following result:

$$f_{el.vac} = \frac{[\alpha_1{}^{*2}-\beta_1{}^{*2}-q^2y^2(\alpha_2{}^{*2}-\beta_2{}^{*2})]^2}{[(\alpha_1{}^*+\beta_1{}^*)^2-q^2y^2(\alpha_2{}^*+\beta_2{}^*)^2]^2}$$

$$\phi_{el.vac} = \frac{-2qy(\alpha_1{}^*\beta_2{}^*-\alpha_2{}^*\beta_1{}^*)}{(\alpha_1{}^*+\beta_1{}^*)^2 - q^2y^2(\alpha_2{}^*+\beta_2{}^*)^2} \qquad /17/$$

The remaining two of eqs. /9/ give the quantity γ. The result has the same form as that of Wang[5] /of course, with the starred polynomials/:

$$\gamma = 2\ln \frac{\alpha_1{}^{*2}-\beta_1{}^{*2}-q^2y^2(\alpha_2{}^{*2}-\beta_2{}^{*2})}{p^{2\delta}(x^2-y^2)^{\delta^2}} \qquad /18/$$

Now we can reconstruct the line element and the general duality-rotated electromagnetic potential:

$$ds^2 = \frac{A^2}{B^2}dt^2 - \frac{1}{c^2}\{\frac{A^2B^2}{p^{8\delta}}(x^2-y^2)^{1-4\delta^2} \cdot (\frac{dx^2}{x^2-1} + \frac{dy^2}{(1-y^2)}) + \frac{B^2}{A^2}(x^2-1)(1-y^2)d\psi^2\} \qquad /19/$$

$$\phi = -2e^{iQ}qy\frac{\alpha_1^{*}\beta_2^{*}-\alpha_2^{*}\beta_1^{*}}{B}$$

where c, q and Q are arbitrary real constants, and

$$A = \alpha_1^{*2}-\beta_1^{*2}-q^2y^2(\alpha_2^{*2}-\beta_2^{*2})$$

$$B = (\alpha_1^{*}+\beta_1^{*})^2 - q^2y^2(\alpha_2^{*}+\beta_2^{*})^2 \qquad /20/$$

These formulae are valid for any positive integer δ, thus the line element /19/ is the Bonnorified counterpart of the whole TS series. For any fixed δ, we take the polynomials α_1, α_2, β_1, and β_2 from Ref. 3, and the starred polynomials are to be constructed by means of /15/.

IV. THE BEHAVIOUR OF THE BONNORIFIED TS SOLUTIONS

In this Section we will deal only with the asymptotical ($r\to\infty$) behaviour of the solutions. The central region contains singularities. /For $\delta=1$, see Ref. 12/

From the definition /12/ on the spheroidal coordinates it can be seen that in the limit $r\to\infty$

$$x\to cr \qquad y\to\cos\vartheta \qquad /21/$$

If $x\to\infty$, A and B appearing in the line element are as follow[13]:

$$A = p^{2\delta}x^{2\delta^2}(1+\mathcal{O}(x^{-2}))$$

$$B = p^{2\delta}x^{2\delta^2}(1+\frac{\delta}{px}+\mathcal{O}(x^{-2})) \qquad /22/$$

Neglecting the $\mathcal{O}(x^{-2})$ terms, the following asymptotic line element is obtained:

$$ds^2 = (1-\frac{2\delta}{px})dt^2 - \frac{1}{c^2}(1+\frac{2\delta}{px})\left[dx^2 + \frac{x^2}{1-y^2}dy^2 + x^2(1-y^2)d\psi^2\right] \qquad /23/$$

whence it can be seen that the solution is the Schwarzschild one for great values of x. The asymptotical form of the transformation between the spheroidal and the Schwarzschild coordinates is as follows:

$$x = \frac{\delta}{p}(\frac{r}{m} - 1) + \mathcal{O}(\frac{1}{r})$$

$$y = \cos\vartheta = \mathcal{O}(\frac{1}{r^2}) \qquad /24/$$

where

$$m = \frac{\delta}{pc}$$

is the mass parameter of the solution. This mass may be produced by both the central masses and the electromagnetic field.

The asymptotic form of the electromagnetic potential in the Schwarzschild coordinates can be written as[13]

$$\phi = -e^{iQ}\,\frac{2m^2q\cos\vartheta}{r^2}(1+\frac{m}{r}) + \mathcal{O}(\frac{1}{r^4}) \qquad /25/$$

This is a dipole field /the r^{-3} term comes from the Schwarzschild metric/, the quadrupole momentum is zero. This fact and the axial symmetry imply that the sources are collinear. The leading term of ϕ has been also obtained by Wang[5], similarly, her conclusion for the case q=0 is also right.

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13. These expressions are deduced from the first four TS solutions, for which Ref. 3 gives explicit forms. Nevertheless, it seems probable that the asymptotic forms given here will remain valid for higher δ-s too.

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